Boltzmann-like and Boltzmann-Fokker-Planck Equations as a Foundation of Behavioral Models

Dirk Helbing

II. Institute for Theoretical Physics
University of Stuttgart, Germany

Abstract

It is shown, that the Boltzmann-like equations allow the formulation of a very general model for behavioral changes. This model takes into account spontaneous (or externally induced) behavioral changes and behavioral changes by pair interactions. As most important social pair interactions imitative and avoidance processes are distinguished. The resulting model turns out to include as special cases many theoretical concepts of the social sciences.

A Kramers-Moyal expansion of the Boltzmann-like equations leads to the Boltzmann-Fokker-Planck equations, which allows the introduction of "social forces" and "social fields". A social field reflects the influence of the public opinion, social norms and trends on behaviorial changes. It is not only given by external factors (the environment) but also by the interactions of the individuals. Variations of the individual behavior are taken into account by diffusion coefficients.

1 Introduction

The methods of statistical physics have shown to be very fruitful in physics, but in the last decades they also have become increasingly important in interdisciplinary research. For example, the *master equation* has found many applications in thermodynamics [1], chemical kinetics [2], laser theory [3] and biology [4]. Moreover, WEIDLICH and HAAG have successfully introduced it for the description of social processes [5, 6] like opinion formation [7], migration [8], agglomeration [9] and settlement processes [10].

Another kind of wide-spread equations are the BOLTZMANN equations, which have been developed for the description of the kinetics of gases [11] and of chemical reactions [12]. However, BOLTZMANN-*like equations* [13] play also an important role for quantitative models in the social sciences: It turns out (cf. sect. 2.2) that the *logistic equation* for the description of limited growth processes [14, 15], the socalled *gravity model* for spatial exchange processes [16], and the *game dynamical equations* modelling competition and cooperation processes [17, 18] are special cases of BOLTZMANN-like equations. Moreover, BOLTZMANN-like models have recently been suggested for avoidance processes of pedestrians [19, 20] and for attitude formation by direct pair interactions of individuals occuring in discussions [19, 21].

In this paper we shall show that BOLTZMANN-like equations and BOLTZMANN-FOKKER-PLANCK equations [13] are suited as a foundation of quantitative behavioral models. For this purpose, we shall proceed in the following way: In section 2 the BOLTZMANN-like equations will be introduced and applied to the description of behavioral changes. The model includes *spontaneous* (or *externally induced*) behavioral changes and behavioral changes by *pair interactions* of individuals. These changes are described by *transition rates*. They reflect the results of mental and psychical processes, which could be simulated with help of OSGOOD and TANNENBAUM's *congruity principle* [22], HEIDER's *balance theory* [23] or FESTINGER's *dissonance theory* [24]. However, it is sufficient for our model to determine the transition rates empirically (sect. 5). The *ansatz* used for the transition rates distinguishes *imitative* and *avoidance processes*, and assumes *utility maximization* of the individuals (sect. 2.1). It is shown, that the resulting BOLTZMANN-like model for imitative processes implies as special cases many generally accepted theoretical approaches in the social sciences (sect. 2.2).

In section 3 a consequent mathematical formulation related to an idea of LEWIN [25] is developed, according to which the behavior of individuals is guided by a *social field*. This formulation is achieved by a KRAMERS-MOYAL expansion of the BOLTZMANN-like equations leading to a kind of *diffusion equations*: the socalled BOLTZMANN-FOKKER-PLANCK *equations* [13]. In these equations the most probable behavioral change is given by a vectorial quantity that can be interpreted as *social force* (sect. 3.1). The social force results from external influences (the environment) as well as from individual interactions. In special cases the social force is the gradient of a potential. This potential reflects the public opinion, social norms and trends, and will be called the *social field*. By *diffusion coefficients* an individual variation of the behavior (the "freedom of will") is taken into account. In section 4 representative cases are illustrated by computer simulations.

The BOLTZMANN-FOKKER-PLANCK modell for the behavior of individuals under the influence of a social field shows some analogies with the physical model for the behavior of

1

electrons in an electric field (e.g. of an atomic nucleus) [13, 19] (cf. HARTREE's *selfconsistent field ansatz* [26]). Especially, individuals and electrons influence the concrete form of the *effective* social resp. electric field. However, the behavior of electrons is governed by a *different* equation: the SCHRÖDINGER *equation*.

In physics, the BOLTZMANN-FOKKER-PLANCK equations can be used for the description of *diffusion* processes [27].

2 The BOLTZMANN-like equations

Let us consider a *system* consisting of a great number $N \gg 1$ of *subsystems*. These subsystems are in one *state* $\boldsymbol{x}$ of several possible states combined in the set Ω.

Due to *fluctuations* one cannot expect a deterministic theory for the temporal change $d\boldsymbol{x}/dt$ of the state $\boldsymbol{x}(t)$ to be realistic. However, one can construct a *stochastic* model for the change of the *probability distribution* $P(\boldsymbol{x}, t)$ of states $\boldsymbol{x}(t)$ within the given system $(P(\boldsymbol{x}, t) \geq 0, \sum_{x \in \Omega} P(\boldsymbol{x}, t) = 1)$. By introducing an index a we may distinguish A different *types* a of subsystems. If N_a denotes the number of subsystems of type a, we have $\sum_{a=1}^{A} N_a = N$, and the following relation holds:

$$P(\boldsymbol{x}, t) = \sum_{a=1}^{A} \frac{N_a}{N} P_a(\boldsymbol{x}, t) \,. \tag{1}$$

Our goal is now to find a suitable equation for the probability distribution $P_a(\boldsymbol{x}, t)$ of states for subsystems of type a $(P_a(\boldsymbol{x}, t) \geq 0, \sum_{x \in \Omega} P_a(\boldsymbol{x}, t) = 1)$. If we neglect memory effects (cf. sect. 6.1), the desired equation has the form of a *master equation* [13, 19]:

$$\frac{d}{dt} P_a(\boldsymbol{x}, t) = \sum_{\substack{x' \in \Omega \\ (x' \neq x)}} \left[w^a(\boldsymbol{x}|\boldsymbol{x}'; t) P_a(\boldsymbol{x}', t) - w^a(\boldsymbol{x}'|\boldsymbol{x}; t) P_a(\boldsymbol{x}, t) \right] \,. \tag{2}$$

$w^a(\boldsymbol{x}'|\boldsymbol{x}; t)$ is the *effective transition rate* from state $\boldsymbol{x}$ to $\boldsymbol{x}'$ and takes into account the fluctuations. Restricting the model to spontaneous (or externally induced) transitions and transitions due to pair interactions, we have [13, 19]:

$$w^a(\boldsymbol{x}'|\boldsymbol{x}; t) := w_a(\boldsymbol{x}'|\boldsymbol{x}; t) + \sum_{b=1}^{A} \sum_{y \in \Omega} \sum_{y' \in \Omega} N_b \, \widetilde{w}_{ab}(\boldsymbol{x}', \boldsymbol{y}'|\boldsymbol{x}, \boldsymbol{y}; t) P_b(\boldsymbol{y}, t) \,. \tag{3}$$

$w_a(\boldsymbol{x}'|\boldsymbol{x}; t)$ describes the rate of spontaneous (resp. externally induced) transitions from $\boldsymbol{x}$ to $\boldsymbol{x}'$ for subsystems of type a. $\widetilde{w}_{ab}(\boldsymbol{x}', \boldsymbol{y}'|\boldsymbol{x}, \boldsymbol{y}; t)$ is the transition rate for two subsystems of types a and b to change their states from $\boldsymbol{x}$ and $\boldsymbol{y}$ to $\boldsymbol{x}'$ and $\boldsymbol{y}'$ due to pair interactions.

Inserting (3) into (2), we now obtain the socalled BOLTZMANN-*like equations* [13, 19]

$$\frac{d}{dt} P_a(\boldsymbol{x}, t) = \sum_{x' \in \Omega} \left[w_a(\boldsymbol{x}|\boldsymbol{x}'; t) P_a(\boldsymbol{x}', t) - w_a(\boldsymbol{x}'|\boldsymbol{x}; t) P_a(\boldsymbol{x}, t) \right] \tag{4a}$$

$$+ \sum_{b=1}^{A} \sum_{x' \in \Omega} \sum_{y \in \Omega} \sum_{y' \in \Omega} w_{ab}(\boldsymbol{x}, \boldsymbol{y'}|\boldsymbol{x'}, \boldsymbol{y}; t) P_b(\boldsymbol{y}, t) P_a(\boldsymbol{x'}, t)$$

$$- \sum_{b=1}^{A} \sum_{x' \in \Omega} \sum_{y \in \Omega} \sum_{y' \in \Omega} w_{ab}(\boldsymbol{x'}, \boldsymbol{y'}|\boldsymbol{x}, \boldsymbol{y}; t) P_b(\boldsymbol{y}, t) P_a(\boldsymbol{x}, t) \qquad (4b)$$

with

$$w_{ab}(\boldsymbol{x'}, \boldsymbol{y'}|\boldsymbol{x}, \boldsymbol{y}; t) := N_b \, \widetilde{w}_{ab}(\boldsymbol{x'}, \boldsymbol{y'}|\boldsymbol{x}, \boldsymbol{y}; t) . \qquad (5)$$

Obviously, (4b) depends nonlinearly on the probability distributions $P_a(\boldsymbol{x}, t)$, which is due to the interaction processes.

Neglecting spontaneous transitions (i.e., $w_a(\boldsymbol{x'}|\boldsymbol{x}; t) \equiv 0$) the BOLTZMANN-like equations agree with the BOLTZMANN equations, that originally have been developed for the description of the kinetics of gases [11]. A more detailed discussion can be found in [19].

In order to apply the BOLTZMANN-like equations to behavioral changes we have now to take the following specifications given by table 1 (cf. [19]):

STATISTICAL PHYSICS	BEHAVIORAL MODELS
system	population
subsystems	individuals
states	behaviors (concerning a special topic of interest)
types of subsystems	types of behavior (subpopulations)
transitions	behavioral changes
interactions	imitative processes, avoidance processes
fluctuations	"freedom of will"

Table 1: Specification of the notions used in statistical physics for an application to behavioral models.

It is possible to generalize the resulting behavioral model to simultaneous interactions of an arbitrary number of individuals (i.e., higher order interactions) [13, 19]. However, in most cases behavioral changes are dominated by pair interactions. Many of the phenomena occuring in social interaction processes can already be understood by the discussion of pair interactions.

2.1 The form of the transition rates

In the following we have to find a concrete form of the effective transition rates

$$w^a(\boldsymbol{x'}|\boldsymbol{x}; t) = w_a(\boldsymbol{x'}|\boldsymbol{x}; t) + \sum_{b=1}^{A} \sum_{y \in \Omega} \sum_{y' \in \Omega} W_{ab}(\boldsymbol{x'}|\boldsymbol{x}, \boldsymbol{y}; t) P_b(\boldsymbol{y}, t) , \qquad (6a)$$

$$W_{ab}(\boldsymbol{x}'|\boldsymbol{x}, \boldsymbol{y}; t) := \sum_{y' \in \Omega} w_{ab}(\boldsymbol{x}', \boldsymbol{y}'|\boldsymbol{x}, \boldsymbol{y}; t) \qquad (6b)$$

(cf. (3)) that is suitable for the description of behavioral changes. $W_{ab}(\boldsymbol{x}'|\boldsymbol{x}, \boldsymbol{y}; t)$ is the rate of pair interactions

$$\boldsymbol{x}' \xleftarrow{\boldsymbol{y}} \boldsymbol{x}, \qquad (7)$$

where an individual of subpopulation a changes the behavior from $\boldsymbol{x}$ to $\boldsymbol{x}'$ under the influence of an individual of subpopulation b showing the behavior $\boldsymbol{y}$. There are only two important kinds of social pair interactions:

$$\boldsymbol{x}' \xleftarrow{\boldsymbol{x}'} \boldsymbol{x} \qquad (\boldsymbol{x}' \neq \boldsymbol{x}) \qquad (8a)$$

$$\boldsymbol{x}' \xleftarrow{\boldsymbol{x}} \boldsymbol{x} \qquad (\boldsymbol{x}' \neq \boldsymbol{x}). \qquad (8b)$$

Obviously, the interpretation of the above kinds of pair interactions is the following:

- The interactions (8a) describe *imitative processes* (processes of persuasion), that means, the tendency to take over the behavior $\boldsymbol{x}'$ of another individual.

- The interactions (8b) describe *avoidance processes*, where an individual changes the behavior when meeting another individual showing the same behavior $\boldsymbol{x}$. Processes of this kind are known as aversive behavior, defiant behavior or snob effect.

The corresponding transition rates are of the general form

$$W_{ab}(\boldsymbol{x}'|\boldsymbol{x}, \boldsymbol{y}; t) \quad := \quad \nu_{ab}(t) R^1_{ab}(\boldsymbol{x}'|\boldsymbol{x}; t) \delta_{yx'} \qquad (9a)$$
$$+ \quad \nu_{ab}(t) R^2_{ab}(\boldsymbol{x}'|\boldsymbol{x}; t) \delta_{yx}, \qquad (9b)$$

where the term (9a) describes imitative processes and the term (9b) describes avoidance processes. δ_{yx} has the meaning of the KRONECKER function. By inserting (9) into (6) we arrive at the following general form of the effective transition rates:

$$w^a(\boldsymbol{x}'|\boldsymbol{x}; t) := \nu_a(t) R_a(\boldsymbol{x}'|\boldsymbol{x}; t) + \sum_{b=1}^A \nu_{ab}(t) \Big[R^1_{ab}(\boldsymbol{x}'|\boldsymbol{x}; t) P_b(\boldsymbol{x}', t) + R^2_{ab}(\boldsymbol{x}'|\boldsymbol{x}; t) P_b(\boldsymbol{x}, t) \Big].$$
$$(10a)$$

For behavioral models one often assumes

$$R^k_{ab}(\boldsymbol{x}'|\boldsymbol{x}; t) := f^k_{ab}(t) R^a(\boldsymbol{x}'|\boldsymbol{x}; t). \qquad (10b)$$

In (10),

- $\nu_a(t)$ is a measure for the rate of spontaneous (or externally induced) behavioral changes within subpopulation a.

- $R_a(\boldsymbol{x}'|\boldsymbol{x}; t)$ [resp. $R^a(\boldsymbol{x}'|\boldsymbol{x}; t)$] is the *readiness* for an individual of subpopulation a to change the behavior from $\boldsymbol{x}$ to $\boldsymbol{x}'$ spontaneously [resp. in pair interactions].

- $\nu_{ab}(t) \equiv N_b \tilde{\nu}_{ab}(t)$ is the *interaction rate* of an individual of subpopulation a with individuals of subpopulation b.

- $f_{ab}^1(t)$ is a measure for the frequency of imitative processes.

- $f_{ab}^2(t)$ is a measure for the frequency of avoidance processes.

A more detailled discussion of the different kinds of interaction processes and of *ansatz* (10) is given in [19, 28].

For $R^a(\boldsymbol{x}'|\boldsymbol{x};t)$ we take the quite general form

$$R^a(\boldsymbol{x}'|\boldsymbol{x};t) = \frac{e^{U^a(x',t)-U^a(x,t)}}{D_a(\boldsymbol{x}',\boldsymbol{x};t)} \tag{11a}$$

with

$$D_a(\boldsymbol{x}',\boldsymbol{x};t) = D_a(\boldsymbol{x},\boldsymbol{x}';t) > 0$$

(cf. [8, 19]). Then, the readiness $R^a(\boldsymbol{x}'|\boldsymbol{x};t)$ for an individual of subpopulation a to change the behavior from $\boldsymbol{x}$ to $\boldsymbol{x}'$ will be the greater,

- the greater the *difference* of the *utilities* $U^a(.,t)$ of behaviors $\boldsymbol{x}'$ and $\boldsymbol{x}$ is,

- the smaller the *incompatibility ("distance")* $D_a(\boldsymbol{x}',\boldsymbol{x};t)$ between the behaviors $\boldsymbol{x}$ and $\boldsymbol{x}'$ is.

Similar to (11a) we use

$$R_a(\boldsymbol{x}'|\boldsymbol{x};t) = \frac{e^{U_a(x',t)-U_a(x,t)}}{D_a(\boldsymbol{x}',\boldsymbol{x};t)} \, , \tag{11b}$$

and, therefore, allow the utility function $U_a(\boldsymbol{x},t)$ for spontaneous (or externally induced) behavioral changes to differ from the utility function $U^a(\boldsymbol{x},t)$ for behavioral changes in pair interactions. *Ansatz* (11) is related to the *multinomial logit model* [29, 30], and assumes *utility maximization* with incomplete information about the exact utility of a behavioral change from $\boldsymbol{x}$ to $\boldsymbol{x}'$, which is, therefore, estimated and stochastically varying (cf. [19]).

Computer simulations of the BOLTZMANN-like equations (2), (10), (11) are discussed and illustrated in [19, 21, 28] (cf. also sect. 4).

2.2 Special fields of application in the social sciences

The BOLTZMANN-like equations (2), (10) include a variety of special cases, which have become very important in the social sciences:

- The *logistic equation* [14, 15] describes limited growth processes. Let us consider the situation of two behaviors $\boldsymbol{x} \in \{1,2\}$ (i.e., $P_a(1,t) = 1 - P_a(2,t)$) and one subpopulation ($A = 1$). $\boldsymbol{x} = 2$ may, for example, have the meaning to apply a certain strategy, and $\boldsymbol{x} = 1$ not to do so. If only imitative processes

$$2 \xleftarrow{2} 1 \tag{12}$$

and processes of spontaneous replacement

$$1 \longleftarrow 2 \tag{13}$$

are considered, one arrives at the *logistic equation*

$$
\begin{aligned}
\frac{d}{dt}P_1(2,t) &= -\nu_1(t)R_1(1|2;t)P_1(2,t) + \nu_{11}(t)f_{11}^1(t)R^1(2|1;t)\big(1 - P_1(2,t)\big)P_1(2,t) \\
&\equiv A(t)P_1(2,t)\big(B(t) - P_1(2,t)\big) .
\end{aligned}
\tag{14}
$$

- The *gravity model* [16] describes processes of exchange between different places $\boldsymbol{x}$. It results for $R_a(\boldsymbol{x}'|\boldsymbol{x};t) \equiv 0$, $f_{ab}^1(t) \equiv 1$, $f_{ab}^2(t) \equiv 0$, and $A = 1$:

$$
\frac{d}{dt}P(\boldsymbol{x},t) = \nu(t) \sum_{x' \in \Omega} \left[\frac{e^{U(x,t)-U(x',t)}}{D(\boldsymbol{x},\boldsymbol{x}')} - \frac{e^{U(x',t)-U(x,t)}}{D(\boldsymbol{x}',\boldsymbol{x})} \right] P(\boldsymbol{x},t)P(\boldsymbol{x}',t) .
\tag{15}
$$

  Here, we have dropped the index a because of $a = 1$. $P(\boldsymbol{x},t)$ is the probability of being at place $\boldsymbol{x}$. The absolute rate of exchange from $\boldsymbol{x}$ to $\boldsymbol{x}'$ is proportional to the probabilities $P(\boldsymbol{x},t)$ and $P(\boldsymbol{x}',t)$ at the places $\boldsymbol{x}$ and $\boldsymbol{x}'$. $D(\boldsymbol{x},\boldsymbol{x}')$ is often chosen as a function of the metric distance $\|\boldsymbol{x} - \boldsymbol{x}'\|$ between $\boldsymbol{x}$ and $\boldsymbol{x}'$: $D(\boldsymbol{x},\boldsymbol{x}') \equiv D(\|\boldsymbol{x}-\boldsymbol{x}'\|)$.

- The *behavioral model* of WEIDLICH and HAAG [5, 6, 8] assumes spontaneous transitions due to *indirect interactions*, which are, for example, induced by the media (TV, radio, or newspapers). We obtain this model for $f_{ab}^1(t) \equiv 0 \equiv f_{ab}^2(t)$ and

$$
U_a(\boldsymbol{x},t) := \delta_a(\boldsymbol{x},t) + \sum_{b=1}^{A} \kappa_{ab}\, P_b(\boldsymbol{x},t) .
\tag{16}
$$

  $\delta_a(\boldsymbol{x},t)$ is the *preference* of subpopulation a for behavior $\boldsymbol{x}$. κ_{ab} are *coupling parameters* describing the influence of the behaviorial distribution within subpopulation b on the behavior of subpoplation a. For $\kappa_{ab} > 0$, κ_{ab} reflects the *social pressure* of behavioral majorities.

- The *game dynamical equations* [17, 18, 19, 31] result for $f_{ab}^1(t) \equiv \delta_{ab}$, $f_{ab}^2(t) \equiv 0$, and

$$
R^a(\boldsymbol{x}'|\boldsymbol{x};t) := \max\big(E_a(\boldsymbol{x}',t) - E_a(\boldsymbol{x},t), 0\big)
\tag{17}
$$

(cf. [19, 28]). Their explicit form is

$$
\begin{aligned}
\frac{d}{dt}P_a(\boldsymbol{x},t) &= \sum_{x' \in \Omega} \Big[w_a(\boldsymbol{x}|\boldsymbol{x}';t)P_a(\boldsymbol{x}',t) - w_a(\boldsymbol{x}'|\boldsymbol{x};t)P_a(\boldsymbol{x},t) \Big] \tag{18a} \\
&\quad + \nu_{aa}(t)P_a(\boldsymbol{x},t)\Big[E_a(\boldsymbol{x},t) - \langle E_a \rangle \Big] . \tag{18b}
\end{aligned}
$$

Whereas (18a) again describes spontaneous behavioral changes (*"mutations"*, innovations), (18b) reflects competition processes leading to a *"selection"* of behaviors with a *success* $E_a(\boldsymbol{x},t)$ that exceeds the *average success*

$$
\langle E_a \rangle := \sum_{x' \in \Omega} E_a(\boldsymbol{x}',t)P_a(\boldsymbol{x}',t) .
\tag{19}
$$

6

The success $E_a(\boldsymbol{x}, t)$ is connected with the socalled *payoff matrices* $\underline{A}_{ab} \equiv \left(A_{ab}(\boldsymbol{x}, \boldsymbol{y})\right)$ by

$$E_a(\boldsymbol{x}, t) := A_a(\boldsymbol{x}) + \sum_{b=1}^{A} \sum_{y \in \Omega} A_{ab}(\boldsymbol{x}, \boldsymbol{y}) P_b(\boldsymbol{y}, t) \tag{20}$$

[19, 28]. $A_a(\boldsymbol{x})$ means the success of behavior $\boldsymbol{x}$ with respect to the environment.

Since the game dynamical equations (18) agree with the *selection mutation equations* [17] they are not only a powerful tool in social sciences and economy [18, 31, 32, 33], but also in evolutionary biology [34, 35, 36, 37].

3 The Boltzmann-Fokker-Planck equations

We shall now assume the set Ω of possible behaviors to build a *continuous* space. The n dimensions of this space correspond to different characteristic *aspects* of the considered behaviors. In the continuous formulation, the sums in (2), (3) have to be replaced by integrals:

$$\begin{aligned} \frac{d}{dt} P_a(\boldsymbol{x}, t) &= \int_{\Omega} d^n x' \left[w^a(\boldsymbol{x}|\boldsymbol{x}'; t) P_a(\boldsymbol{x}', t) - w^a(\boldsymbol{x}'|\boldsymbol{x}; t) P_a(\boldsymbol{x}, t) \right] \\ &= \int d^n x' \left[w^a[\boldsymbol{x}'|\boldsymbol{x} - \boldsymbol{x}'; t] P_a(\boldsymbol{x} - \boldsymbol{x}', t) - w^a[\boldsymbol{x}'|\boldsymbol{x}; t] P_a(\boldsymbol{x}, t) \right], \end{aligned} \tag{21a}$$

where

$$w^a[\boldsymbol{x}' - \boldsymbol{x}|\boldsymbol{x}; t] := w^a(\boldsymbol{x}'|\boldsymbol{x}; t) := w_a(\boldsymbol{x}'|\boldsymbol{x}; t) + \sum_{b=1}^{A} \int_{\Omega} d^n y \int_{\Omega} d^n y' \, N_b \, \widetilde{w}_{ab}(\boldsymbol{x}', \boldsymbol{y}'|\boldsymbol{x}, \boldsymbol{y}; t) P_b(\boldsymbol{y}, t). \tag{21b}$$

A reformulation of the Boltzmann-like equations (21) via a Kramers-Moyal expansion [38, 39] (second order Taylor approximation) leads to a kind of *diffusion equations*: the socalled Boltzmann-Fokker-Planck *equations* [13]

$$\frac{\partial}{\partial t} P_a(\boldsymbol{x}, t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big[K_{ai}(\boldsymbol{x}, t) P_a(\boldsymbol{x}, t) \Big] + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Big[Q_{aij}(\boldsymbol{x}, t) P_a(\boldsymbol{x}, t) \Big] \tag{22a}$$

with the effective *drift coefficients*

$$K_{ai}(\boldsymbol{x}, t) := \int_{\Omega} d^n x' \, (x_i' - x_i) w^a(\boldsymbol{x}'|\boldsymbol{x}; t) \tag{22b}$$

and the effective *diffusion coefficients*[1]

$$Q_{aij}(\boldsymbol{x}, t) := \int_{\Omega} d^n x' \, (x_i' - x_i)(x_j' - x_j) w^a(\boldsymbol{x}'|\boldsymbol{x}; t). \tag{22c}$$

---

[1]In [13] the expression for $Q_{aij}(\boldsymbol{x}, t)$ contains additional terms due to another derivation of (22). However, they make no contributions, since they result in vanishing surface integrals (cf. [19]).

Whereas the drift coefficients $K_{ai}(\boldsymbol{x}, t)$ govern the systematic change of the distribution $P_a(\boldsymbol{x}, t)$, the diffusion coefficients $Q_{aij}(\boldsymbol{x}, t)$ describe the spread of the distribution $P_a(\boldsymbol{x}, t)$ due to fluctuations resulting from the individual variation of behavioral changes.

For *ansatz* (10), the effective drift and diffusion coefficients can be splitted into contributions due to spontaneous (or externally induced) transitions ($k = 0$), imitative processes ($k = 1$), and avoidance processes ($k = 2$):

$$K_{ai}(\boldsymbol{x}, t) = \sum_{k=0}^{2} K_{ai}^{k}(\boldsymbol{x}, t), \qquad Q_{aij}(\boldsymbol{x}, t) = \sum_{k=0}^{2} Q_{aij}^{k}(\boldsymbol{x}, t), \tag{23a}$$

where

$$
\begin{aligned}
K_{ai}^{0}(\boldsymbol{x}, t) &:= \nu_a(t) \int d^n x' \, (x_i' - x_i) R_a(\boldsymbol{x}'|\boldsymbol{x}; t), \\
K_{ai}^{1}(\boldsymbol{x}, t) &:= \sum_{b=1}^{A} \nu_{ab}(t) f_{ab}^{1}(t) \int d^n x' \, (x_i' - x_i) R^a(\boldsymbol{x}'|\boldsymbol{x}; t) P_b(\boldsymbol{x}', t), \\
K_{ai}^{2}(\boldsymbol{x}, t) &:= \sum_{b=1}^{A} \nu_{ab}(t) f_{ab}^{2}(t) \int d^n x' \, (x_i' - x_i) R^a(\boldsymbol{x}'|\boldsymbol{x}; t) P_b(\boldsymbol{x}, t)
\end{aligned}
\tag{23b}
$$

and

$$
\begin{aligned}
Q_{aij}^{0}(\boldsymbol{x}, t) &:= \nu_a(t) \int d^n x' \, (x_i' - x_i)(x_j' - x_j) R_a(\boldsymbol{x}'|\boldsymbol{x}; t), \\
Q_{aij}^{1}(\boldsymbol{x}, t) &:= \sum_{b=1}^{A} \nu_{ab}(t) f_{ab}^{1}(t) \int d^n x' \, (x_i' - x_i)(x_j' - x_j) R^a(\boldsymbol{x}'|\boldsymbol{x}; t) P_b(\boldsymbol{x}', t), \\
Q_{aij}^{2}(\boldsymbol{x}, t) &:= \sum_{b=1}^{A} \nu_{ab}(t) f_{ab}^{2}(t) \int d^n x' \, (x_i' - x_i)(x_j' - x_j) R^a(\boldsymbol{x}'|\boldsymbol{x}; t) P_b(\boldsymbol{x}, t).
\end{aligned}
\tag{23c}
$$

The behavioral changes induced by the *environment* are included in $K_{ai}^{0}(\boldsymbol{x}, t)$ and $Q_{aij}^{0}(\boldsymbol{x}, t)$.

3.1 Social force and social field

The BOLTZMANN-FOKKER-PLANCK equations (22a) are equivalent to the stochastic equations (LANGEVIN equations)

$$\frac{dx_i}{dt} = F_{ai}(\boldsymbol{x}, t) + \sum_{j=1}^{n} G_{aij}(\boldsymbol{x}, t) \xi_j(t) \tag{24a}$$

with

$$K_{ai}(\boldsymbol{x}, t) = F_{ai}(\boldsymbol{x}, t) + \frac{1}{2} \sum_{j,k=1}^{n} \left[\frac{\partial}{\partial x_k} G_{aij}(\boldsymbol{x}, t) \right] G_{ajk}(\boldsymbol{x}, t) \tag{24b}$$

and

$$Q_{aij}(\boldsymbol{x}, t) = \sum_{k=1}^{n} G_{aik}(\boldsymbol{x}, t) G_{akj}(\boldsymbol{x}, t) \tag{24c}$$

(cf. [19]). For an individual of subpopulation a the vector $\boldsymbol{\zeta}_a(\boldsymbol{x}, t)$ with the components

$$\zeta_{ai}(\boldsymbol{x}, t) = \sum_{j=1}^{n} G_{aij}(\boldsymbol{x}, t)\xi_j(t) \tag{25}$$

describes the contribution to the change of behavior $\boldsymbol{x}$ that is caused by behavioral fluctuations $\boldsymbol{\xi}(t)$ (which are assumed to be delta-correlated and GAUSSian [19]). Since the diffusion coefficients $Q_{aij}(\boldsymbol{x}, t)$ and the coefficients $G_{aij}(\boldsymbol{x}, t)$ are usually small quantities, we have $F_{ai}(\boldsymbol{x}, t) \approx K_{ai}(\boldsymbol{x}, t)$ (cf. (24b)), and (24a) can be put into the form

$$\frac{d\boldsymbol{x}}{dt} \approx \boldsymbol{K}_a(\boldsymbol{x}, t) + \text{fluctuations.} \tag{26}$$

Whereas the fluctuation term describes individual behavioral variations, the vectorial quantity $\boldsymbol{K}_a(\boldsymbol{x}, t)$ drives the systematic change of the behavior $\boldsymbol{x}(t)$ of individuals of subpopulation a. Therefore, it is justified to denote $\boldsymbol{K}_a(\boldsymbol{x}, t)$ as *social force* acting on individuals of subpopulation a.

The social force influences the behavior of the individuals, but, conversely, due to interactions, the behavior of the individuals also influences the social force via the behavioral distributions $P_a(\boldsymbol{x}, t)$ (cf. (21b), (22b)). That means, $K_a(\boldsymbol{x}, t)$ is a function of the social processes within the given population.

Under the integrability conditions

$$\frac{\partial}{\partial x_j} K_{ai}(\boldsymbol{x}, t) = \frac{\partial}{\partial x_i} K_{aj}(\boldsymbol{x}, t) \qquad \text{for all } i, j \tag{27}$$

there exists a time-dependent *potential*

$$V_a(\boldsymbol{x}, t) := -\int^{x} d\boldsymbol{x}' \cdot \boldsymbol{K}_a(\boldsymbol{x}', t), \tag{28}$$

so that the social force is given by its gradient ∇:

$$\boldsymbol{K}_a(\boldsymbol{x}, t) = -\nabla V_a(\boldsymbol{x}, t). \tag{29}$$

The potential $V_a(\boldsymbol{x}, t)$ can be understood as *social field*. It reflects the social influences and interactions relevant for behavioral changes: the public opinion, trends, social norms, etc.

3.2 Discussion of the concept of force

Clearly, the social force is no force obeying the NEWTONian laws of mechanics. Instead, the social force $\boldsymbol{K}_a(\boldsymbol{x}, t)$ is a vectorial quantity with the following properties:

- $\boldsymbol{K}_a(\boldsymbol{x}, t)$ drives the temporal change $d\boldsymbol{x}/dt$ of another vectorial quantity: the behavior $\boldsymbol{x}(t)$ of an individual of subpopulation a.

- The component

$$\boldsymbol{K}_{ab}(\boldsymbol{x},t) := \nu_{ab}(t) \int_\Omega d^n x' \, (\boldsymbol{x}' - \boldsymbol{x}) \Big[f_{ab}^1(t) P_b(\boldsymbol{x}',t) + f_{ab}^2(t) P_b(\boldsymbol{x},t) \Big] R^a(\boldsymbol{x}'|\boldsymbol{x};t) \quad (30)$$

of the social force $\boldsymbol{K}_a(\boldsymbol{x},t)$ describes the reaction of subpopulation a on the behavioral distribution within subpopulation b and usually differs from $\boldsymbol{K}_{ba}(\boldsymbol{x},t)$, which describes the influence of subpopulation a on subpopulation b.

- Neglecting fluctuations, the behavior $\boldsymbol{x}(t)$ does not change if $\boldsymbol{K}_a(\boldsymbol{x},t)$ vanishes. $\boldsymbol{K}_a(\boldsymbol{x},t) = \boldsymbol{0}$ corresponds to an *extremum* of the social field $V_a(\boldsymbol{x},t)$, because it means

$$\nabla V_a(\boldsymbol{x},t) = \boldsymbol{0} \,. \quad (31)$$

We can now formulate our results in the following form related to LEWIN's *"field theory"* [25]:

- Let us assume that an individuals objective is to behave in an optimal way with respect to the social field, that means, he or she tends to a behavior corresponding to a *minimum* of the social field.

- If the behavior $\boldsymbol{x}$ does not agree with a minimum of the social field $V_a(\boldsymbol{x},t)$ this evokes a *psychical tension (force)*

$$\boldsymbol{K}_a(\boldsymbol{x},t) = -\nabla V_a(\boldsymbol{x},t) \quad (32)$$

that is given by the gradient of the social field $V_a(\boldsymbol{x},t)$.

- The psychical tension $\boldsymbol{K}_a(\boldsymbol{x},t)$ is a vectorial quantity that induces a behavioral change according to

$$\frac{d\boldsymbol{x}}{dt} \approx \boldsymbol{K}_a(\boldsymbol{x},t) \,. \quad (33)$$

- The behavioral change $d\boldsymbol{x}/dt$ drives the behavior $\boldsymbol{x}(t)$ towards a minimum $\boldsymbol{x}_a^*$ of the social field $V_a(\boldsymbol{x},t)$.

- When the behavior has reached a minimum $\boldsymbol{x}_a^*$ of the social field $V_a(\boldsymbol{x},t)$, it holds

$$\nabla V_a(\boldsymbol{x},t) = \boldsymbol{0} \quad (34)$$

and, therefore, $\boldsymbol{K}_a(\boldsymbol{x},t) = \boldsymbol{0}$, that means, the psychical tension vanishes.

- If the psychical tension $\boldsymbol{K}_a(\boldsymbol{x},t)$ vanishes, except for fluctuations no behavioral changes take place—in accordance with (33).

In the special case, where an individual's objective is the behavior $\boldsymbol{x}_a^*$, one would expect behavioral changes according to

$$\frac{d\boldsymbol{x}}{dt} \approx \gamma(\boldsymbol{x}_a^* - \boldsymbol{x}) \,, \quad (35)$$

10

which corresponds to a social field

$$V_a(\boldsymbol{x}, t) \approx \frac{\gamma}{2}(\boldsymbol{x}_a^* - \boldsymbol{x})^2 \tag{36}$$

with a minimum at $\boldsymbol{x}_a^*$. Examples for this case are discussed in [40].

Note, that the social fields $V_a(\boldsymbol{x}, t)$ of different subpopulations a usually have *different* minima $\boldsymbol{x}_a^*$. That means, individuals of different subpopulations a will normally feel different psychical tensions $\boldsymbol{K}_a(\boldsymbol{x}, t)$. This shows the psychical tension $\boldsymbol{K}_a(\boldsymbol{x}, t)$ to be a *"subjective"* quantity.

4 Computer simulations

In the following, the BOLTZMANN-FOKKER-PLANCK equations for behavioral changes will be illustrated by representative computer simulations. We shall examine the case of $A = 2$ subpopulations, and situations for which the interesting aspect of the individual behavior can be described by a certain *position* $x \in [1/20, 1]$ on a one-dimensional continuous scale (i.e., $n = 1$, $\boldsymbol{x} \equiv x$). Then, the integrability conditions (27) are automatically fulfilled, and the social field

$$V_a(x, t) = -\int_{x_0}^{x} dx' \, K_a(x', t) - c_a(t) \tag{37}$$

is well-defined. The parameter $c_a(t)$ can be chosen arbitrarily. We will take for $c_a(t)$ the value that shifts the absolute minimum of $V_a(x, t)$ to zero, that means,

$$c_a(t) := \min_x \left(-\int_{x_0}^{x} dx' \, K_a(x', t) \right) . \tag{38}$$

- Since we will restrict the simulations to the case of imitative or avoidance processes, the shape of the social field $V_a(x, t)$ changes with time only due to changes of the probability distributions $P_a(x, t)$ (cf. (23)), that means, due to behavioral changes of the individuals (see figures 1 to 6).

In the one-dimensional case one can find the formal *stationary solution*

$$P_a(x) = P_a(x_0)\frac{Q_a(x_0)}{Q_a(x)} \exp \left(2\int_{x_0}^{x} dx' \, \frac{K_a(x')}{Q_a(x')} \right) , \tag{39}$$

which we expect to be approached in the limit of large times $t \to \infty$. Due to the dependence of $K_a(x)$ and $Q_a(x)$ on $P_a(x)$, equations (39) are only *implicit* equations. However, from (39) we can derive the following conclusions:

- If the diffusion coefficients are constant ($Q_a(x) \equiv Q_a$), (39) simplifies to

$$P_a(x) = P_a(x_0) \exp \left(-\frac{2}{Q_a}\big[V_a(x) + c_a\big] \right) , \tag{40}$$

11

that means, the stationary solution $P_a(x)$ is completely determined by the social field $V_a(x)$. Especially, $P_a(x)$ has its maxima at the positions x_a^*, where the social field $V_a(x)$ has its minima (see fig. 1). The diffusion constant Q_a regulates the width of the behavioral distribution $P_a(x)$.

- If the diffusion coefficients $Q_a(x)$ are varying functions of the position x, the behavioral distribution $P_a(x)$ is also influenced by the concrete form of $Q_a(x)$. From (39) one expects high behavioral probabilities $P_a(x)$ where the diffusion coefficients $Q_a(x)$ are small (see fig. 2, where the probability distribution $P_1(x)$ cannot be explained solely by the social field $V_1(x)$).

- Since the stationary solution $P_a(x)$ depends on both, $K_a(x)$ and $Q_a(x)$, different combinations of $K_a(x)$ and $Q_a(x)$ can lead to the same probability distribution $P_a(x)$ (see fig. 4 in the limit of large times).

For the following simulations, we shall use the *ansatz*

$$R^a(x'|x;t) = \frac{e^{U^a(x',t)-U^a(x,t)}}{D_a(x',x;t)} \tag{41a}$$

for the readiness $R^a(x'|x;t)$ to change from x to x' (cf. (11a)). With the utility function

$$U^a(x,t) := -\frac{1}{2}\left(\frac{x-x_a}{l_a}\right)^2, \qquad l_a := \frac{L_a}{20} \tag{41b}$$

subpopulation a prefers behavior x_a. L_a means the *indifference* of subpopulation a with respect to variations of the position x. Moreover, we take

$$\frac{\nu_{ab}(t)}{D_a(x',x;t)} := e^{-|x'-x|/r}, \qquad r = \frac{R}{20}, \tag{41c}$$

where R can be interpreted as measure for the *range of interaction*. According to (41c), the rate of behavioral changes is the smaller the greater they are. Only small changes of the position (i.e., between neighboring positions) contribute with an appreciable rate.

4.1   Sympathy and interaction frequency

Let $s_{ab}(t)$ be the degree of *sympathy* which individuals of subpopulation a feel towards individuals of subpopulation b. Then, one expects the following: Whereas the frequency $f_{ab}^1(t)$ of imitative processes will be increasing with $s_{ab}(t)$, the frequency $f_{ab}^2(t)$ of avoidance processes will be decreasing with $s_{ab}(t)$. This functional relationship can, for example, be described by

$$\begin{aligned} f_{ab}^1(t) &:= f_a^1(t)\, s_{ab}(t)\,, \\ f_{ab}^2(t) &:= f_a^2(t)\Big(1 - s_{ab}(t)\Big) \end{aligned} \tag{42}$$

with

$$0 \le s_{ab}(t) \le 1\,. \tag{43}$$

$f_a^1(t)$ is a measure for the frequency of imitative processes within subpopulation a, $f_a^2(t)$ a measure for the frequency of avoidance processes. If we assume the sympathy between individuals of the same subpopulation to be be maximal, we have $s_{11}(t) \equiv 1 \equiv s_{22}(t)$.

4.2 Imitative processes ($f_a^1(t) \equiv 1$, $f_a^2(t) \equiv 0$)

In the following simulations of imitative processes we assume the prefered positions to be $x_1 = 6/20$ and $x_2 = 15/20$. With

$$\left(s_{ab}(t)\right) \equiv \left(f_{ab}^1(t)\right) := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{44}$$

the individuals of subpopulation $a = 1$ like the individuals of subpopulation $a = 2$, but not the other way round. That means, subpopulation 2 influences subpopulation 1, but not vice versa. One could say, the individuals of subpopulation 2 act as *trendsetters*.

As expected, in both behavioral distributions $P_a(x,t)$ there appears a maximum around the prefered behavior x_a. In addition, due to imitative processes of subpopulation 1, a second maximum of $P_1(x,t)$ develops around the prefered behavior x_2 of the trendsetters. This second maximum is small, if the indifference L_1 of subpopulation 1 with respect to variations of the position x is low (see fig. 1). For high values of the indifference L_1 even the *majority* of individuals of subpopulation 1 imitates the behavior of the trendsetters (see fig. 2)!

We shall now consider the case

$$\left(s_{ab}(t)\right) \equiv \left(f_{ab}^1(t)\right) := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{45}$$

for which the subpopulations influence each other mutually with equal strengths. If the indifference L_a with respect to changes of the position x is small in both subpopulations a, *each* probability distribution $P_a(x,t)$ has *two* maxima. The higher maximum is located around the prefered position x_a. A second maximum can be found around the position prefered in the *other* subpopulation. It is the higher, the greater the indifference L_a is (see fig. 3).

However, if L_a exceeds a certain value in at least one subpopulation, only *one* maximum develops in each behavioral distribution $P_a(x,t)$! Despite the fact, that the social fields $V_a(x,t)$ and diffusion coefficients $Q_a(x,t)$ of the subpopulations a are different because of their different prefered positions x_a (and different utility functions $U^a(x,t)$), the behavioral distributions $P_a(x,t)$ agree after some time! Especially, the maxima x_a^* of the distributions $P_a(x,t)$ are located at the *same* position x^* in both subpopulations. x^* is nearer to the position x_a of the subpopulation a with the lower indifference L_a (see fig. 4).

4.3 Avoidance processes ($f_a^1(t) \equiv 0$, $f_a^2(t) \equiv 1$)

For the simulation of avoidance processes we assume with $x_1 = 9/20$ and $x_2 = 12/20$ that both subpopulations nearly prefer the same behavior. Figure 5 shows the case, where the individuals of different subpopulations dislike each other:

$$\left(s_{ab}(t)\right) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{i.e.,} \qquad \left(f_{ab}^2(t)\right) \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{46}$$

This corresponds to a mutual influence of one subpopulation on the respective other. The computational results prove:

- The individuals avoid behaviors which can be found in the other subpopulation.

- The subpopulation $a = 1$ with the lower indifference $L_1 < L_2$ is distributed around the prefered behavior x_1 and pushes away the other subpopulation!

Despite the fact that the initial behavioral distribution $P_a(x, 0)$ agrees in both subpopulations, there is nearly no overlapping of $P_1(x, t)$ and $P_2(x, t)$ after some time. This is a typical example for *polarization phenomena* in the society.

In figure 6, we assume that the individuals of subpopulation 2 like the individuals of subpopulation 1 and, therefore, do not react on the behaviors in subpopulation 1 with avoidance processes:

$$\left(s_{ab}(t) \right) := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{i.e.,} \qquad \left(f_{ab}^2(t) \right) \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{47}$$

As a consequence, $P_2(x, t)$ remains unchanged with time, whereas $P_1(x, t)$ drifts away from the prefered behavior x_1 due to avoidance processes. Surprisingly, the polarization effect is much smaller than in figure 5! The distributions $P_1(x, t)$ and $P_2(x, t)$ overlap considerably. This is, because the slope of $P_2(x, t)$ is smaller than in figure 5 (and remains constant). As a consequence, the probability for an individual of subpopulation 1 to meet a disliked individual of subpopulation 2 with the same behavior x can hardly be decreased by a small behavioral change. One may conclude, that polarization effects (which often lead to an escalation) can be reduced, if individuals do not return dislike by dislike.

5 Empirical determination of the model parameters

For practical purposes one has, of course, to determine the model parameters from empirical data. Therefore, let us assume to know empirically the distribution functions $P_a^e(\boldsymbol{x}, t_l)$, [the interaction rates $\nu_{ab}^e(t_l)$] and the effective transition rates $w_e^a(\boldsymbol{x}'|\boldsymbol{x}; t_l)$ $(\boldsymbol{x}' \neq \boldsymbol{x})$ for a couple of times $t_l \in \{t_0, \ldots, t_L\}$. The corresponding effective social fields $V_a^e(\boldsymbol{x}, t_l)$ and diffusion coefficients $Q_{aij}^e(\boldsymbol{x}, t_l)$ are, then, easily obtained as

$$V_a^e(\boldsymbol{x}, t_l) := - \int^{\boldsymbol{x}} d\boldsymbol{x}' \cdot \boldsymbol{K}_a^e(\boldsymbol{x}', t_l) \tag{48a}$$

with

$$K_{ai}^e(\boldsymbol{x}, t_l) := \int_\Omega d^n x' \, (x_i' - x_i) w_e^a(\boldsymbol{x}'|\boldsymbol{x}; t_l), \tag{48b}$$

and

$$Q_{aij}^e(\boldsymbol{x}, t_l) := \int_\Omega d^n x' \, (x_i' - x_i)(x_j' - x_j) w_e^a(\boldsymbol{x}'|\boldsymbol{x}; t_l). \tag{49}$$

Much more difficult is the determination of the utility functions $U_a^e(\boldsymbol{x}, t_l)$, $U_e^a(\boldsymbol{x}, t_l)$, the distance functions $D_a^e(\boldsymbol{x}', \boldsymbol{x}; t_l)$, and the rates $\nu_a^e(t_l)$, $\nu_{ab}^{1e}(t_l) := \nu_{ab}^e(t_l) f_{ab}^{1e}(t_l)$, $\nu_{ab}^{2e}(t_l) := \nu_{ab}^e(t_l) f_{ab}^{2e}(t_l)$. This can be done by numerical minimization of the *error function*

$$F := \sum_{a=1}^A \sum_{l=0}^L \sum_{\substack{x, x' \in \Omega \\ (x' \neq x)}} \frac{1}{2} \left\{ \left[w_e^a(\boldsymbol{x}'|\boldsymbol{x}; t_l) - \frac{1}{D_a(\boldsymbol{x}', \boldsymbol{x}; t_l)} g_a(\boldsymbol{x}', \boldsymbol{x}; t_l) \right] P_a^e(\boldsymbol{x}, t_l) \right\}^2, \tag{50}$$

for example with the method of *steepest descent* [41]. In (50), we have introduced the abbreviation

$$g_a(\boldsymbol{x}', \boldsymbol{x}; t_l) := \nu_a(t_l) \mathrm{e}^{U_a(x', t_l) - U_a(x, t_l)} + \sum_{b=1}^{A} \left[\nu_{ab}^1(t_l) P_b^\mathrm{e}(\boldsymbol{x}', t_l) + \nu_{ab}^2(t_l) P_b^\mathrm{e}(\boldsymbol{x}, t_l) \right] \mathrm{e}^{U^a(x', t_l) - U^a(x, t_l)}.$$
(51)

It turns out (cf. [19]), that the rates $\nu_a(t_l)$ have to be taken constant during the minimization process (e.g., $\nu_a(t_l) \equiv 1$), whereas the parameters $U_a(\boldsymbol{x}, t_l)$, $U^a(\boldsymbol{x}, t_l)$, $\nu_{ab}^1(t_l) := \nu_{ab}^\mathrm{e}(t_l) f_{ab}^1(t_l)$ and $\nu_{ab}^2(t_l) := \nu_{ab}^\mathrm{e}(t_l) f_{ab}^2(t_l)$ are to be varied. For $1/D_a(\boldsymbol{x}', \boldsymbol{x}; t_l)$ one inserts

$$\frac{1}{D_a(\boldsymbol{x}', \boldsymbol{x}; t_l)} = \frac{n_a(\boldsymbol{x}', \boldsymbol{x}; t_l)}{d_a(\boldsymbol{x}', \boldsymbol{x}; t_l)}$$
(52a)

with

$$n_a(\boldsymbol{x}', \boldsymbol{x}; t_l) := w_\mathrm{e}^a(\boldsymbol{x}'|\boldsymbol{x}; t_l) g_a(\boldsymbol{x}', \boldsymbol{x}; t_l) \left[P_a^\mathrm{e}(\boldsymbol{x}, t_l) \right]^2 + w_\mathrm{e}^a(\boldsymbol{x}|\boldsymbol{x}'; t_l) g_a(\boldsymbol{x}, \boldsymbol{x}'; t_l) \left[P_a^\mathrm{e}(\boldsymbol{x}', t_l) \right]^2$$
(52b)

and

$$d_a(\boldsymbol{x}', \boldsymbol{x}; t_l) := \left[g_a(\boldsymbol{x}', \boldsymbol{x}; t_l) P_a^\mathrm{e}(\boldsymbol{x}, t_l) \right]^2 + \left[g_a(\boldsymbol{x}, \boldsymbol{x}'; t_l) P_a^\mathrm{e}(\boldsymbol{x}', t_l) \right]^2.$$
(52c)

(52) follows from the minimum condition for $D_a(\boldsymbol{x}', \boldsymbol{x}; t_l)$ (cf. [19]).

Since F may have a couple of minima due to its nonlinearity, suitable start parameters have to be taken. Especially, the numerically determined rates $\nu_{ab}^1(t_l)$ and $\nu_{ab}^2(t_l)$ have to be non-negative.

If F is minimal for the parameters $U_a(\boldsymbol{x}, t_l)$, $U^a(\boldsymbol{x}, t_l)$, $D_a(\boldsymbol{x}', \boldsymbol{x}; t_l)$, $\nu_a(t_l)$, $\nu_{ab}^1(t_l)$ and $\nu_{ab}^2(t_l)$, this is (as can easily be checked) also true for the scaled parameters

$$\begin{aligned}
U_a^\mathrm{e}(\boldsymbol{x}, t_l) &:= U_a(\boldsymbol{x}, t_l) - C_a(t_l), \\
U_\mathrm{e}^a(\boldsymbol{x}, t_l) &:= U^a(\boldsymbol{x}, t_l) - C^a(t_l), \\
D_a^\mathrm{e}(\boldsymbol{x}', \boldsymbol{x}; t_l) &:= \frac{D_a(\boldsymbol{x}', \boldsymbol{x}; t_l)}{D_a(t_l)}, \\
\nu_a^\mathrm{e}(t_l) &:= \frac{\nu_a(t_l)}{D_a(t_l)}, \\
\nu_{ab}^{1\mathrm{e}}(t_l) &:= \frac{\nu_{ab}^1(t_l)}{D_a(t_l)}, \\
\nu_{ab}^{2\mathrm{e}}(t_l) &:= \frac{\nu_{ab}^2(t_l)}{D_a(t_l)}.
\end{aligned}$$
(53)

In order to obtain unique results we put

$$\sum_{x \in \Omega} U_a^\mathrm{e}(\boldsymbol{x}, t_l) \stackrel{!}{\equiv} 0, \qquad \sum_{x \in \Omega} U_\mathrm{e}^a(\boldsymbol{x}, t_l) \stackrel{!}{\equiv} 0,$$
(54)

and

$$\sum_{\substack{x, x' \in \Omega \\ (x' \neq x)}} \frac{1}{D_a^\mathrm{e}(\boldsymbol{x}', \boldsymbol{x}; t_l)} \stackrel{!}{\equiv} \sum_{\substack{x, x' \in \Omega \\ (x' \neq x)}} 1,$$
(55)

which leads to

$$C_a(t_l) := \frac{\sum\limits_{x \in \Omega} U_a(\boldsymbol{x}, t_l)}{\sum\limits_{x \in \Omega} 1} \,, \qquad C^a(t_l) := \frac{\sum\limits_{x \in \Omega} U^a(\boldsymbol{x}, t_l)}{\sum\limits_{x \in \Omega} 1} \,, \tag{56}$$

and

$$\frac{1}{D_a(t_l)} := \frac{\sum\limits_{\substack{x,x' \in \Omega \\ (x' \neq x)}} \dfrac{1}{D_a(\boldsymbol{x}', \boldsymbol{x}; t_l)}}{\sum\limits_{\substack{x,x' \in \Omega \\ (x' \neq x)}} 1} \,. \tag{57}$$

$C_a(t_l)$ and $C^a(t_l)$ are *mean utilities*, whereas $D_a(t_l)$ is a kind of *unit of distance*.

The distances $D_a^{\mathrm{e}}(\boldsymbol{x}', \boldsymbol{x}; t)$ are suitable quantities for *multidimensional scaling* [42, 43]. They reflect the "psychical structure" (psychical topology) of individuals of subpopulation a, since they determine which behaviors are more or less related (compatible). By the dependence on a, $D_a^{\mathrm{e}}(\boldsymbol{x}', \boldsymbol{x}; t)$ distinguishes different psychical structures resulting in different types a of behavior and, therefore, different "characters".

6   Summary and outlook

In this article, a behavioral model has been proposed that incorporates in a consistent way many models of social theory: the diffusion models, the multinomial logit model, LEWIN's field theory, the logistic equation, the gravity model, the WEIDLICH-HAAG model, and the game dynamical equations. This very general model opens new perspectives concerning a theoretical description and understanding of behavioral changes, since it is formulated fully mathematically. It takes into account spontaneous (or externally induced) behavioral changes and behavioral changes due to pair interactions. Two important kinds of pair interactions have been distinguished: imitative processes and avoidance processes. The model turns out to be suitable for computational simulations, but it can also be applied to concrete empirical data.

6.1   Memory effects

The formulation of the model in the previous sections has neglected *memory effects* that may also influence behavioral changes. However, memory effects can be easily included by generalizing the BOLTZMANN-like equations to

$$\frac{d}{dt} P_a(\boldsymbol{x}, t) = \int_{t_0}^{t} dt' \sum_{x' \in \Omega} \left[w_{t-t'}^a(\boldsymbol{x}|\boldsymbol{x}'; t') P_a(\boldsymbol{x}', t') - w_{t-t'}^a(\boldsymbol{x}'|\boldsymbol{x}; t') P_a(\boldsymbol{x}, t') \right] \tag{58a}$$

with the effective transition rates

$$w_{t-t'}^a(\boldsymbol{x}'|\boldsymbol{x}; t') := w_a^{t-t'}(\boldsymbol{x}'|\boldsymbol{x}; t') + \sum_{b=1}^{A} \sum_{y \in \Omega} \sum_{y' \in \Omega} w_{ab}^{t-t'}(\boldsymbol{x}', \boldsymbol{y}'|\boldsymbol{x}, \boldsymbol{y}; t') P_b(\boldsymbol{y}, t') \,, \tag{58b}$$

and generalizing the BOLTZMANN-FOKKER-PLANCK equations to

$$
\begin{aligned}
\frac{\partial}{\partial t} P_a(\boldsymbol{x}, t) \;=\; \int_{t_0}^{t} dt' \Bigg\{ &- \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big[K_{ai}^{t-t'}(\boldsymbol{x}, t') P_a(\boldsymbol{x}, t') \Big] \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Big[Q_{aij}^{t-t'}(\boldsymbol{x}, t') P_a(\boldsymbol{x}, t') \Big] \Bigg\}
\end{aligned}
\tag{59a}
$$

with the effective drift coefficients

$$
K_{ai}^{t-t'}(\boldsymbol{x}, t') := \int_{\Omega} d^n x' \, (x_i' - x_i) w_{t-t'}^{a}(\boldsymbol{x}' | \boldsymbol{x}; t') \,,
\tag{59b}
$$

the effective diffusion coefficients

$$
Q_{aij}^{t-t'}(\boldsymbol{x}, t') := \int_{\Omega} d^n x' \, (x_i' - x_i)(x_j' - x_j) w_{t-t'}^{a}(\boldsymbol{x}' | \boldsymbol{x}; t') \,,
\tag{59c}
$$

and

$$
w_{t-t'}^{a}(\boldsymbol{x}' | \boldsymbol{x}; t') := w_a^{t-t'}(\boldsymbol{x}' | \boldsymbol{x}; t') + \sum_{b=1}^{A} \int_{\Omega} d^n y \int_{\Omega} d^n y' \, w_{ab}^{t-t'}(\boldsymbol{x}', \boldsymbol{y}' | \boldsymbol{x}, \boldsymbol{y}; t') P_b(\boldsymbol{y}, t') \,.
\tag{59d}
$$

Obviously, in these formulas there only appears an additional integration over past times t' [19]. The influence of the past results in a dependence of $w_{t-t'}^{a}(\boldsymbol{x}' | \boldsymbol{x}; t')$, $K_{ai}^{t-t'}(\boldsymbol{x}, t')$, and $Q_{aij}^{t-t'}(\boldsymbol{x}, t')$ on $(t - t')$. The BOLTZMANN-like equations (4) resp. the BOLTZMANN-FOKKER-PLANCK equations (22) used in the previous sections result from (58) resp. (59) in the MARKOVian limit

$$
w_{t-t'}^{a}(\boldsymbol{x}' | \boldsymbol{x}; t') := w^{a}(\boldsymbol{x}' | \boldsymbol{x}; t) \delta(t - t')
\tag{60}
$$

of short memory (where $\delta(.)$ is the DIRAC delta function).

6.2   Analogies with chemical reactions

The BOLTZMANN-like equations (4) can also be used for the description of chemical reactions, where the states $\boldsymbol{x}$ denote the different sorts of molecules (or atoms), and a distinguishes different isotopes or conformeres. Imitative and avoidance processes correspond in chemistry to self-activatory and self-inhibitory reactions. Although the concrete transition rates will be different from (10), (11a) in detail, there may be found analogous results for chemical reactions. Note, that the ARRHENIUS formula for the rate of chemical reactions [44] can be put into a form similar to (11a) [19].

References

[1] R. Zwanzig. On the identity of three generalized master equations. *Physica* **30**, 1109–1123, 1964.

[2] I. Oppenheim, K. E. Schuler and G. H. Weiss, eds. *Stochastic Processes in Chemical Physics: The Master Equation.* MIT Press, Cambridge, Mass., 1977.

[3] H. Haken. *Laser Theory.* Springer, Berlin, 1984.

[4] L. Arnold and R. Lefever, eds. *Stochastic Nonlinear Systems in Physics, Chemistry and Biology.* Springer, Berlin, 1981.

[5] W. Weidlich & G. Haag. *Concepts and Models of a Quantitative Sociology. The Dynamics of Interacting Populations.* Springer, Berlin, 1983.

[6] W. Weidlich. Physics and social science—The approach of synergetics. *Physics Reports* **204**, 1–163, 1991.

[7] W. Weidlich. The use of statistical models in sociology. *Collective Phenomena* **1**, 51-59, 1972.

[8] W. Weidlich & G. Haag, eds. *Interregional Migration.* Springer, Berlin, 1988.

[9] W. Weidlich & G. Haag. A dynamic phase transition model for spatial agglomeration processes. *Journal of Regional Science* **27**(4), 529–569, 1987.

[10] W. Weidlich & M. Munz. Settlement formation, Part I: A dynamic theory. *Annals of Regional Science* **24**, 83–106, 1990.

[11] L. Boltzmann. *Lectures on Gas Theory.* University of California, Berkeley, 1964.

[12] F. Wilkinson *Chemical Kinetics and Reaction Mechanisms.* Van Nostrand Reinhold Co., New York, 1980.

[13] D. Helbing. Interrelations between stochastic equations for systems with pair interactions. *Physica A* **181**, 29–52, 1992.

[14] R. Pearl. *Studies in Human Biology.* Williams & Wilkins, Baltimore, 1924.

[15] P. F. Verhulst. *Nuov. Mem. Acad. Roy. Bruxelles* **18**, 1, 1845.

[16] G. K. Zipf. The P1P2/D hypothesis on the intercity movement of persons. *American Sociological Review* **11**, 677–686, 1946.

[17] J. Hofbauer & K. Sigmund. *The Theory of Evolution and Dynamical Systems*. Cambridge University Press, Cambridge, 1988.

[18] P. Schuster, K. Sigmund, J. Hofbauer, R. Wolff, R. Gottlieb & Ph. Merz. Selfregulation of behavior in animal societies I–III. *Biol. Cybern.* **40**, 1–25, 1981.

[19] D. Helbing. *Stochastische Methoden, nichtlineare Dynamik und quantitative Modelle sozialer Prozesse*. PhD thesis, University of Stuttgart, 1992, *submitted to* Oldenbourg publishers, München.

[20] D. Helbing. A fluid dynamic model for the movement of pedestrians. Submitted to *Complex Systems*.

[21] D. Helbing. A mathematical model for attitude formation by pair interactions. *Behavioral Science* **37**, 190–214, 1992.

[22] Ch. E. Osgood & P. H. Tannenbaum. The principle of congruity in the prediction of attitude change. *Psychological Review* **62**, 42–55, 1955.

[23] F. Heider. Attitudes and cognitive organization. *Journal of Psychology* **21**, 107–112, 1946.

[24] L. Festinger. *A Theory of Cognitive Dissonance*. Row & Peterson, Evanston, IL, 1957.

[25] K. Lewin. *Field Theory in Social Science*. Harper & Brothers, New York, 1951.

[26] D. R. Hartree. *Proc. Cambridge philos. Soc.* **24**, 111, 1928.

[27] D. Montgomery. Brownian motion from Boltzmann's equation. *The Physics of Fluids* **14**(10), 2088–2090, 1971.

[28] D. Helbing. A mathematical model for behavioral changes by pair interactions and its relation to game theory. *Angewandte Sozialforschung* **17** (3/4), 179–194, 1991/92.

[29] Th. A. Domencich & D. McFadden. *Urban Travel Demand. A Behavioral Analysis*, pp. 61–69. North-Holland, Amsterdam, 1975.

[30] J. de D. Ortúzar & L. G. Willumsen. *Modelling Transport*. Wiley, Chichester, 1990.

[31] D. Helbing. Stochastic and Boltzmann-like models for behavioral changes, and their relation to game theory. *Physica A*, 1993 (in press).

[32] R. Axelrod. *The Evolution of Cooperation*. Basic Books, New York, 1984.

[33] J. von Neumann & O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, 1944.

[34] M. Eigen. The selforganization of matter and the evolution of biological macromolecules. *Naturwissenschaften* **58**, 465, 1971.

[35] R. A. Fisher. *The Genetical Theory of Natural Selection*. Oxford University Press, Oxford, 1930.

[36] M. Eigen & P. Schuster. *The Hypercycle.* Springer, Berlin, 1979.

[37] R. Feistel & W. Ebeling. *Evolution of Complex Systems.* Kluwer Academic, Dordrecht, 1989.

[38] H. A. Kramers. *Physica* **7**, 284, 1940.

[39] J. E. Moyal. *J. R. Stat. Soc.* **11**, 151–210, 1949.

[40] D. Helbing. A mathematical model for the behavior of pedestrians. *Behavioral Science* **36**, 298–310, 1991.

[41] G. E. Forsythe, M. A. Malcolm & C. B. Moler. *Computer Methods for Mathematical Computations.* Prentice Hall, Englewood Cliffs, N.J., 1977.

[42] J. B. Kruskal & M. Wish. *Multidimensional Scaling.* Sage, Beverly Hills, 1978.

[43] F. W. Young & R. M. Hamer. *Multidimensional Scaling: History, Theory, and Applications.* Lawrence Erlbaum Associates, Hillsdale, N.J., 1987.

[44] C. W. Gardiner. *Handbook of Stochastic Methods.* Springer, Berlin, 2nd edition, 1985.
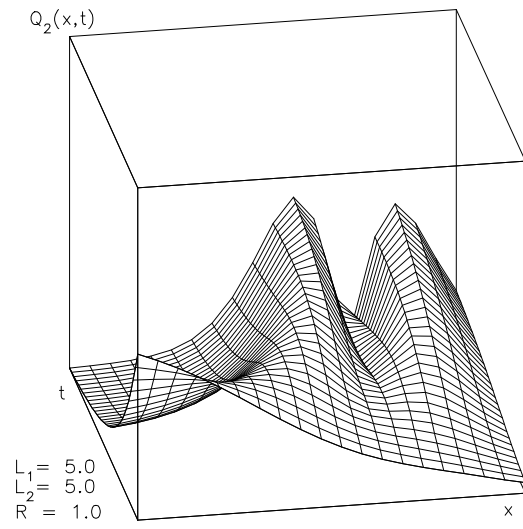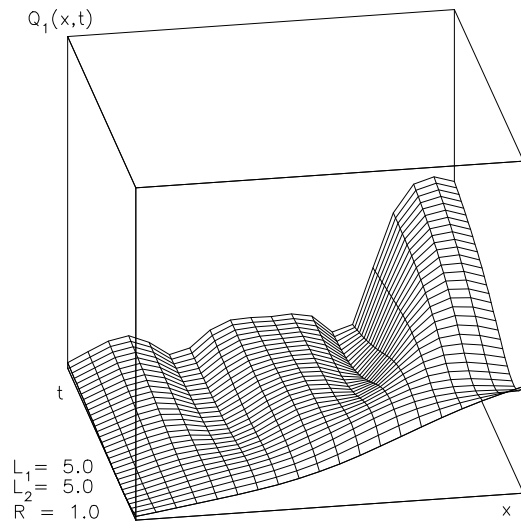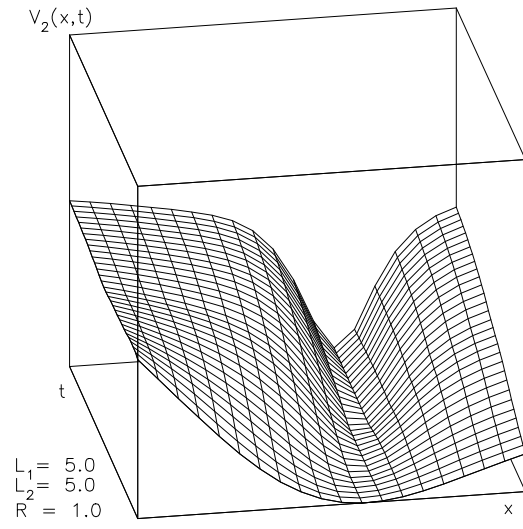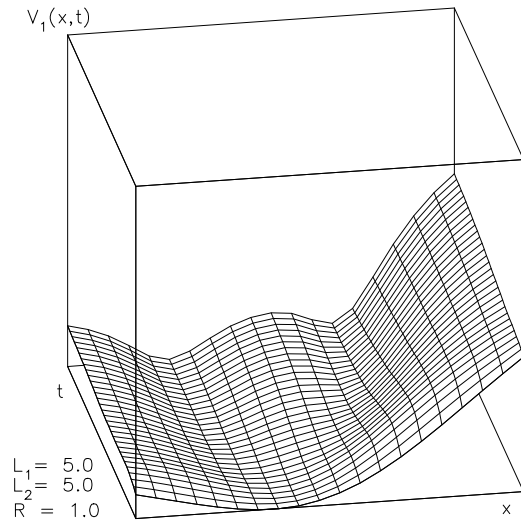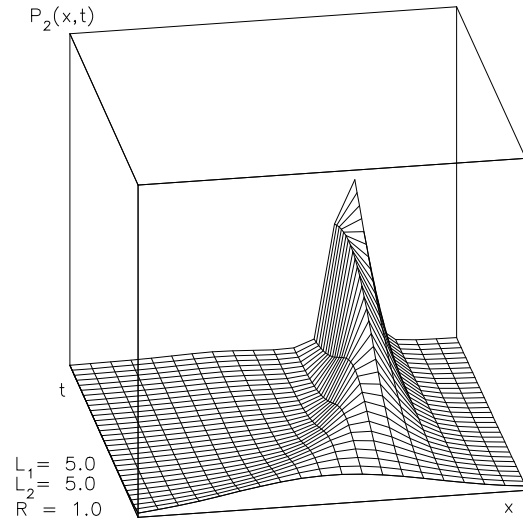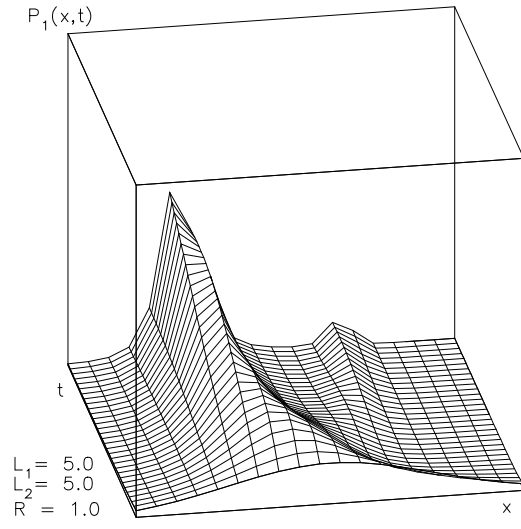
Figure 1: Imitative processes in the case of one-sided sympathy and low indifference L_a with respect to behavioral changes.
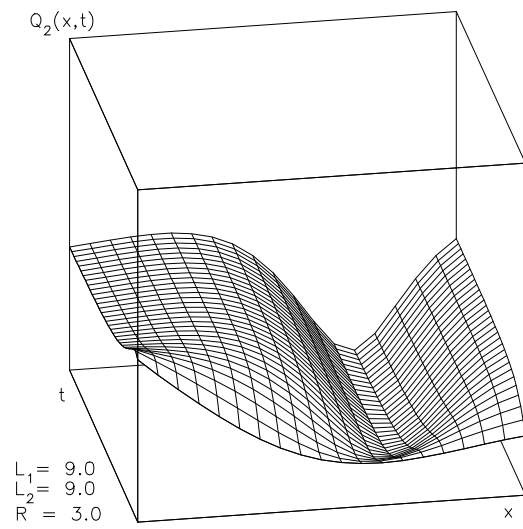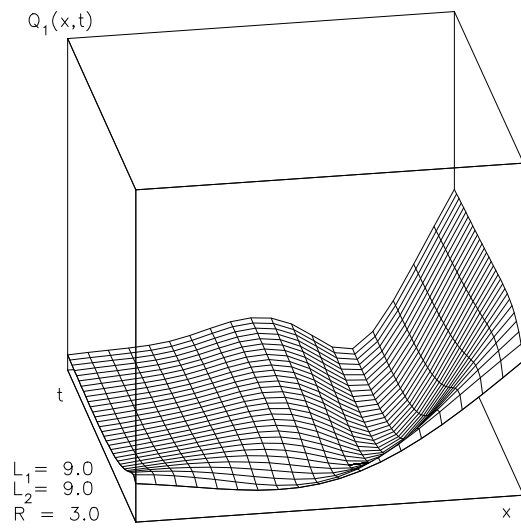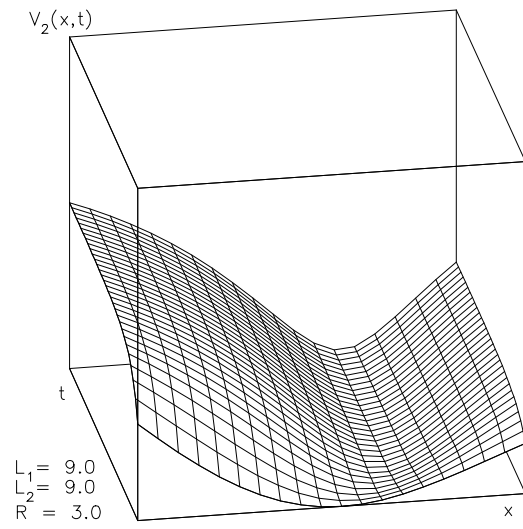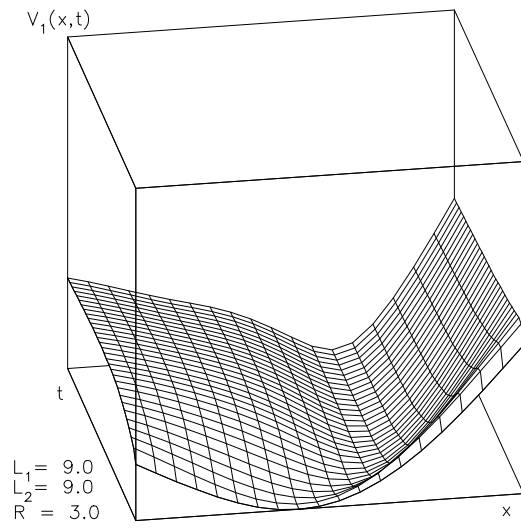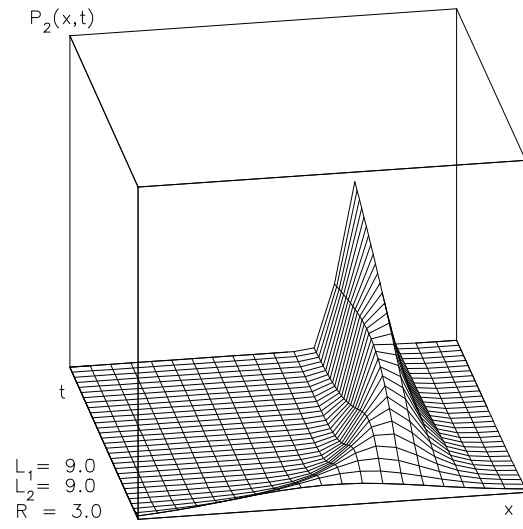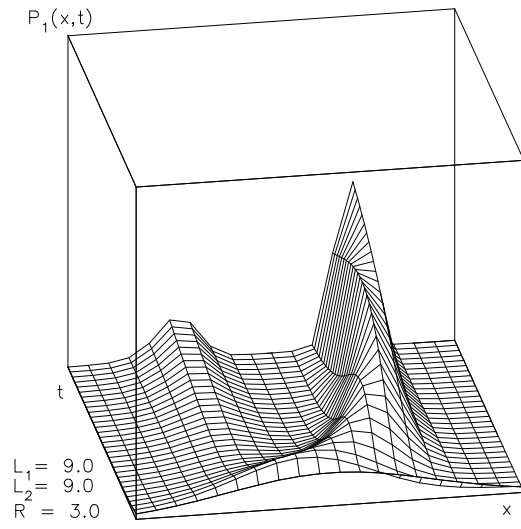
Figure 2: As figure 1, but for high indifference L_a with respect to behavioral changes.
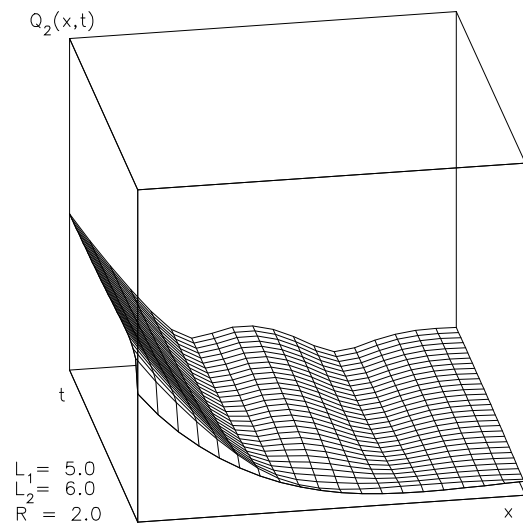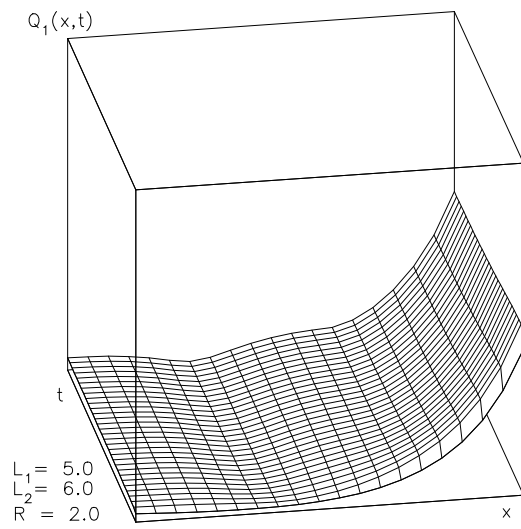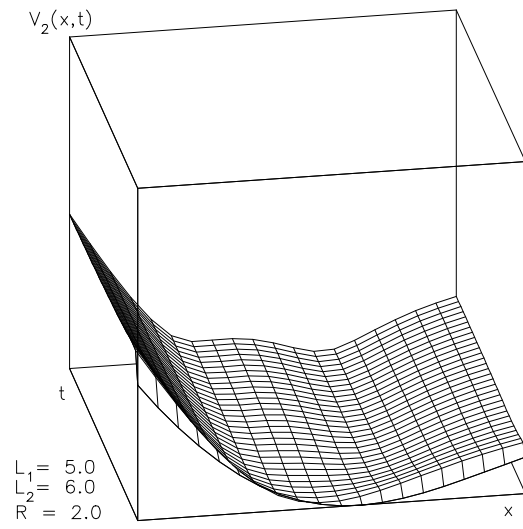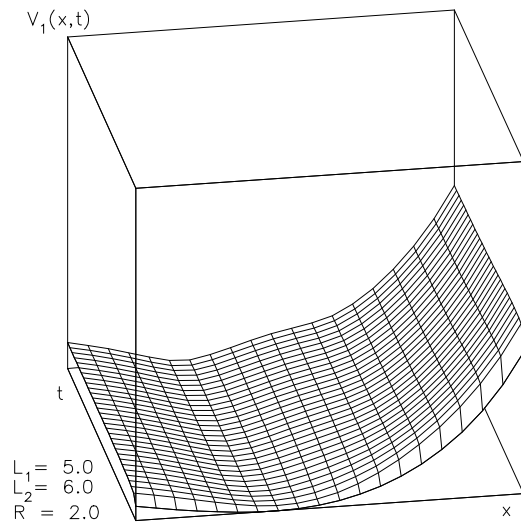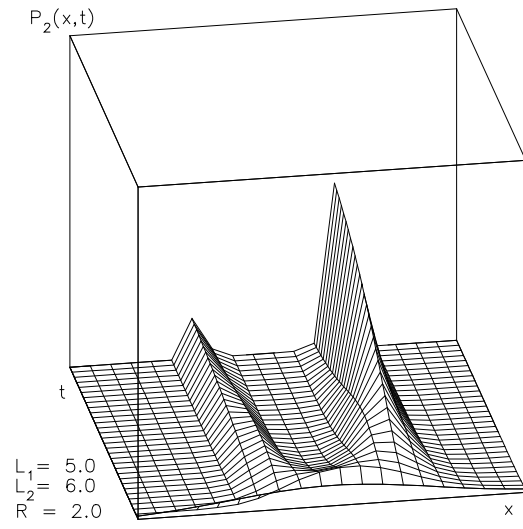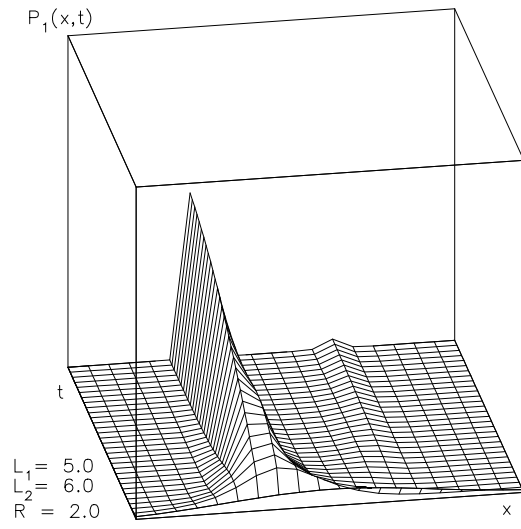
Figure 3: Imitative processes for mutual sympathy and low indifference L_a in both subpopulations.
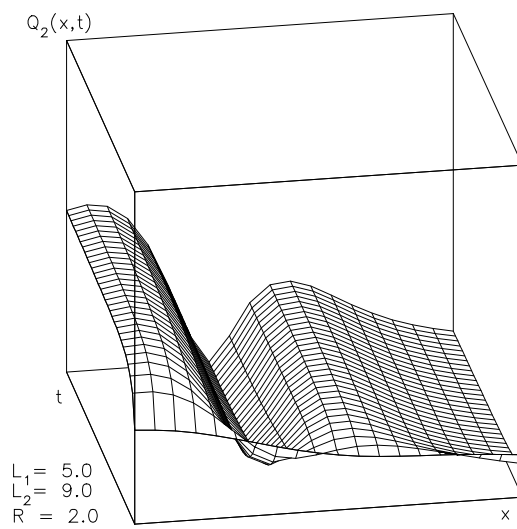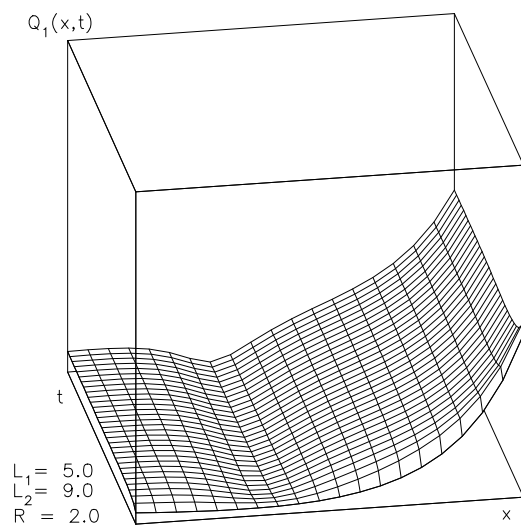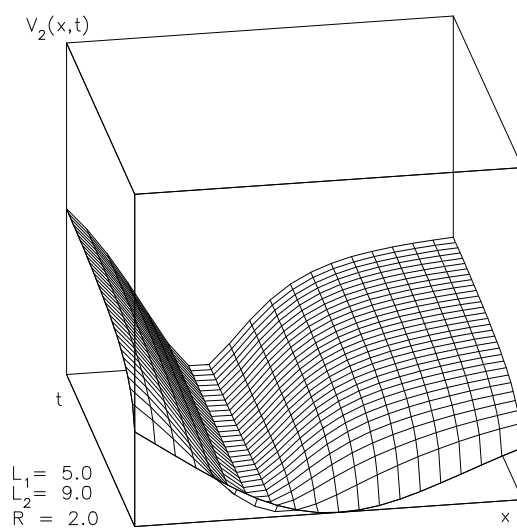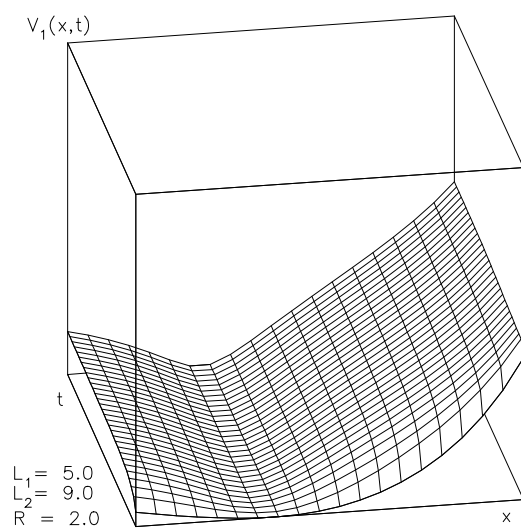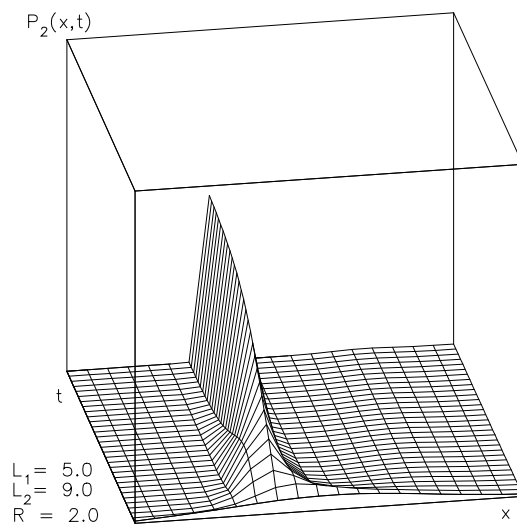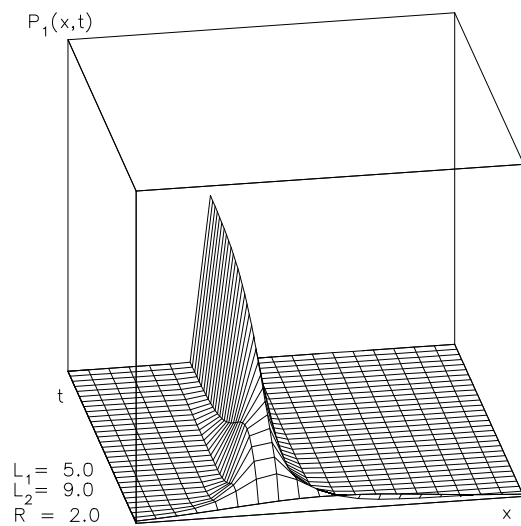
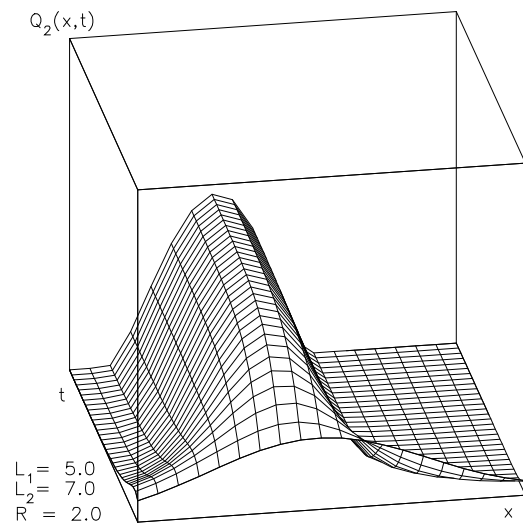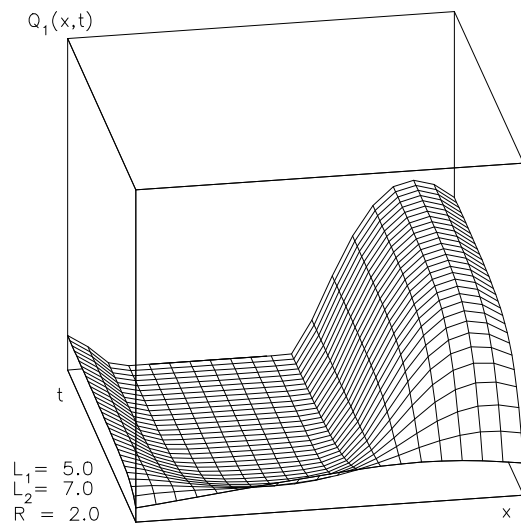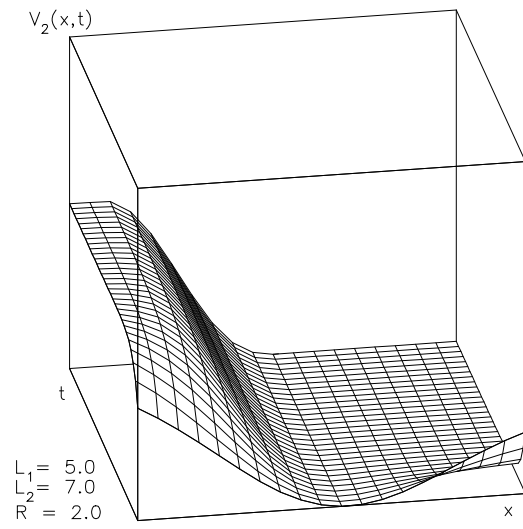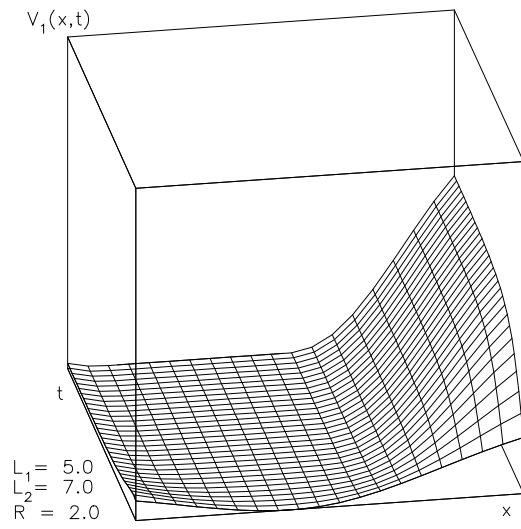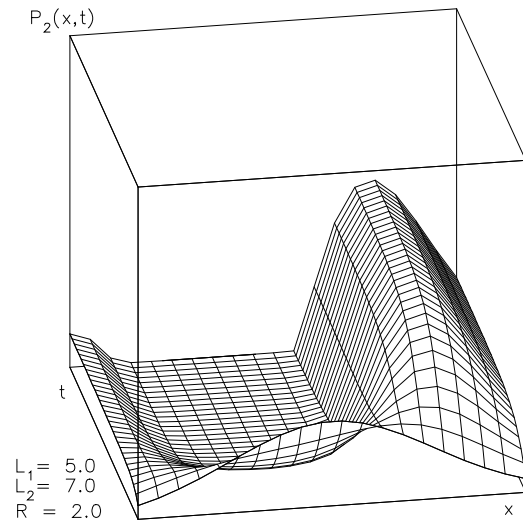Figure 4: As figure 3, but for high indifference L_2 in subpopulation 2.
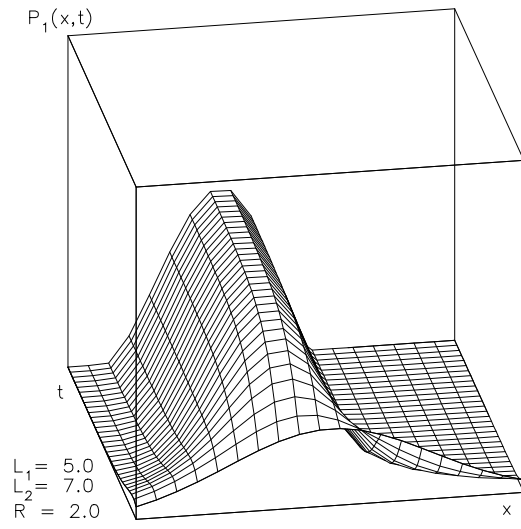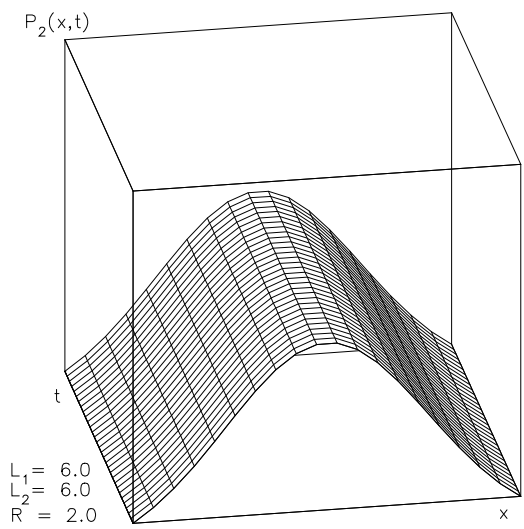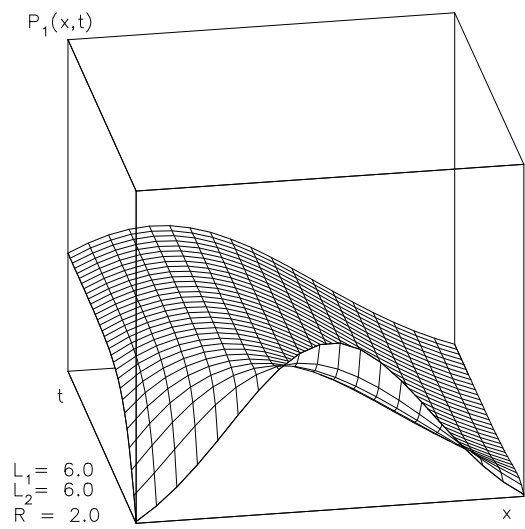
Figure 5: Avoidance processes for mutual dislike of both subpopulations.

Figure 6: Avoidance processes for one-sided dislike.